

Deformation quantization modules on complex symplectic manifolds

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ABSTRACT. We study modules over the algebroid stack $\mathcal{W}_{\mathfrak{X}}$ of deformation quantization on a complex symplectic manifold \mathfrak{X} and recall some results: construction of an algebra for \star -products, existence of (twisted) simple modules along smooth Lagrangian submanifolds, perversity of the complex of solutions for regular holonomic $\mathcal{W}_{\mathfrak{X}}$ -modules, finiteness and duality for the composition of “good” kernels. As a corollary, we get that the derived category of good $\mathcal{W}_{\mathfrak{X}}$ -modules with compact support is a Calabi-Yau category. We also give a conjectural Riemann-Roch type formula in this framework.

Introduction

Let X be a complex manifold, T^*X its cotangent bundle. The conic sheaf of \mathbb{C} -algebras \mathcal{E}_{T^*X} of microdifferential operators on T^*X has been constructed functorially by Sato-Kashiwara-Kawai in [25]. This algebra is associated with the homogeneous symplectic structure and is also naturally defined on the projective cotangent bundle P^*X .

Another (no more conic) algebra on T^*X , denoted here by \mathcal{W}_{T^*X} and defined over a subfield \mathbf{k} of $\mathbb{C}[[\tau^{-1}, \tau]]$ has been constructed in [24] (see [5] for related constructions). Its formal version has been considered by many authors after [1] and extended to Poisson manifolds in [22].

In general, neither the algebras \mathcal{E}_{P^*X} glue on a complex contact manifold, nor the algebras \mathcal{W}_{T^*X} glue on a complex symplectic manifold, although the categories of modules on these non existing algebras make sense. Indeed, one has to replace the notion of a sheaf of algebras by that of an algebroid stack, similarly as one replaces the notion of a sheaf by that of a stack. These constructions are performed in [15], [21], [24] (see also [7] for recent developments and [4, 30, 31] for an algebraic approach).

Here, we start by briefly recalling the constructions of the sheaves \mathcal{E}_{T^*X} and \mathcal{W}_{T^*X} as well as a new sheaf of algebras on T^*X containing \mathcal{W}_{T^*X} , invariant by quantized symplectic transformations, in which the \star -exponential is well defined (see [10]). Then we consider a complex symplectic manifold \mathfrak{X} , introduce the algebroid

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stack $\mathcal{W}_{\mathfrak{X}}$ of deformation quantization on \mathfrak{X} and discuss some recent results on $\mathcal{W}_{\mathfrak{X}}$ -modules:

- If Λ is a smooth Lagrangian submanifold of \mathfrak{X} , there exist twisted simple $\mathcal{W}_{\mathfrak{X}}$ -modules along Λ , the twist being associated with a square root of the line bundle Ω_{Λ} (see [9]).
- Let \mathcal{L}_0 and \mathcal{L}_1 be two regular holonomic modules supported by smooth Lagrangian submanifolds Λ_0 and Λ_1 . Then the complex $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1)$ is a perverse sheaf over the field \mathbf{k} (see [20]).
- Let \mathfrak{X}_i ($i = 1, 2, 3$) be complex symplectic manifolds and denote by \mathfrak{X}_i^a the symplectic manifold deduced from \mathfrak{X}_i by taking the opposite symplectic form. Let \mathcal{K}_i be a good $\mathcal{W}_{\mathfrak{X}_{i+1} \times \mathfrak{X}_i^a}$ -module ($i = 1, 2$) (good means coherent and endowed with a good filtration on each compact subset of \mathfrak{X}) and assume that a properness condition is satisfied by the supports of these modules. Then their composition $\mathcal{K}_2 \circ \mathcal{K}_1$ is a good $\mathcal{W}_{\mathfrak{X}_3 \times \mathfrak{X}_1^a}$ -module. Moreover, composition of kernels commutes with duality (see [28]). As a particular case, we obtain that the triangulated category consisting of good $\mathcal{W}_{\mathfrak{X}}$ -modules with compact supports is Ext-finite over the field \mathbf{k} and admits a Serre functor, namely the shift by $d_{\mathfrak{X}} := \dim_{\mathbb{C}} \mathfrak{X}$.
- The Hochschild homology of the algebroid stack $\mathcal{W}_{\mathfrak{X}}$ is concentrated in degree $-\dim_{\mathbb{C}} \mathfrak{X}$ and is isomorphic to $\mathbf{k}_{\mathfrak{X}}$. This allows us to construct the Euler class $\text{Eu}(\mathcal{M}) \in H_{\text{supp } \mathcal{M}}^{d_{\mathfrak{X}}}(\mathfrak{X}; \mathbf{k}_{\mathfrak{X}})$ of a coherent $\mathcal{W}_{\mathfrak{X}}$ -module. We conjecture that in the situation above, $\text{Eu}(\mathcal{K}_2 \circ \mathcal{K}_1)$ is the relative integral of the cup products $\text{Eu}(\mathcal{K}_1) \cup \text{Eu}(\mathcal{K}_2)$ (see [28]).

This paper summarizes various joint works with A. D'Agnolo [9], G. Dito [10], M. Kashiwara [20], P. Polesello [24] and J-P. Schneiders [28].

1. Microdifferential operators on cotangent bundles

Let X be a complex manifold, $\pi: T^*X \rightarrow X$ its cotangent bundle.

The ring \mathcal{E}_{T^*X} . The manifold T^*X is a complex *homogeneous* symplectic manifold, *i.e.*, T^*X is endowed with a canonical 1-form α_X such that $d\alpha_X$ is symplectic. On T^*X there exists a conic (*i.e.*, constant on the orbits of the action of \mathbb{C}^\times) sheaf \mathcal{E}_{T^*X} constructed functorially by Sato-Kashiwara-Kawai [25] (see also [16, 26] for an exposition) which plays the role of a noncommutative localization of the ring \mathcal{D}_X of differential operators. The sheaf \mathcal{E}_{T^*X} enjoys the following properties:

- \mathcal{E}_{T^*X} is a filtered sheaf of central \mathbb{C} -algebras and

$$\text{gr } \mathcal{E}_{T^*X} \simeq \bigoplus_{j \in \mathbb{Z}} \mathcal{O}_{T^*X}(j).$$

($\mathcal{O}_{T^*X}(j)$ is the subsheaf of \mathcal{O}_{T^*X} consisting of homogeneous functions of degree j in the fibers of π .)

- There is a flat monomorphism of filtered rings $\pi^{-1}\mathcal{D}_X \hookrightarrow \mathcal{E}_{T^*X}$.
- Denote by $\Omega_X^{\frac{1}{2}}$ the twisted sheaf of holomorphic half-forms of maximal degree on X (the notion of twisted sheaves will be recalled below). One defines the sheaf of algebras:

$$(1.1) \quad \mathcal{E}_{T^*X}^{\sqrt{v}} := \pi^{-1}\Omega_X^{\frac{1}{2}} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{E}_{T^*X} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\Omega_X^{-\frac{1}{2}}.$$

(Note that $\mathcal{E}_{T^*X}^{\sqrt{v}}$ is a sheaf although $\Omega_X^{\frac{1}{2}}$ is not a sheaf but a twisted sheaf.) Denote by $a: T^*X \rightarrow T^*X$ the antipodal map, $(x; \xi) \mapsto (x; -\xi)$. There exists a \mathbb{C} -linear anti-isomorphism of sheaves of algebras¹ $a_*\mathcal{E}_{T^*X}^{\sqrt{v}} \xrightarrow{\sim} \mathcal{E}_{T^*X}^{\sqrt{v}}$ called the transposition and denoted $P \mapsto {}^tP$.

- Consider a \mathbb{C}^\times -homogeneous symplectic isomorphism $\varphi: T^*X \supset U \xrightarrow{\sim} V \subset T^*Y$. Then φ can be *locally* quantized as an isomorphism of filtered sheaf of rings commuting with the transposition

$$\Phi: \varphi_*\mathcal{E}_{T^*X}^{\sqrt{v}} \xrightarrow{\sim} \mathcal{E}_{T^*Y}^{\sqrt{v}}.$$

Denote by V^a the image of V by the antipodal map a on T^*Y and by $\Lambda_\varphi \subset U \times V^a$ the image of the graph of φ . This is a Lagrangian submanifold of $U \times V^a$. Locally, we may assume that Ω_X is trivial and there exists an ideal \mathcal{I}_φ of $\mathcal{E}_{T^*(X \times Y)}$ whose associated graded ideal is reduced and coincides with the defining ideal of Λ_φ . Then, for each $\mathcal{E}_{T^*X} \ni P$ there exists a unique $Q \in \mathcal{E}_{T^*Y}$ such that $P - Q \in \mathcal{I}_\varphi$. The correspondence $P \mapsto Q$ is an anti-isomorphism of \mathbb{C} -algebras $\varphi_*\mathcal{E}_{T^*X} \xrightarrow{\sim} a_*\mathcal{E}_{T^*Y}$. One gets the isomorphism Φ by composing with the transposition in \mathcal{E}_{T^*Y} . One shall be aware that this isomorphism exists *only locally and is not unique* in general.

Moreover, when X is affine (*i.e.*, open in some n -dimensional complex vector space), \mathcal{E}_{T^*X} satisfies:

- any section $P \in \mathcal{E}_{T^*X}(U)$ on an open subset $U \subset T^*X$ admits a total symbol

$$(1.2) \quad \sigma_{\text{tot}}(P)(x; \xi) = \sum_{-\infty < j \leq m} p_j(x; \xi), \quad m \in \mathbb{Z} \quad p_j \in \mathcal{O}_{T^*X}(j)(U),$$

with the condition:

$$(1.3) \quad \begin{cases} \text{for any compact subset } K \text{ of } U \text{ there exists a positive constant } C_K \\ \text{such that } \sup_K |p_j| \leq C_K^{-j}(-j)! \text{ for all } j \leq 0. \end{cases}$$

- The total symbol of the product is given by the Leibniz rule:

$$(1.4) \quad \sigma_{\text{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{\text{tot}}(P) \partial_x^\alpha \sigma_{\text{tot}}(Q).$$

- The total symbol of the transposition is given by

$$(1.5) \quad \sigma_{\text{tot}}({}^tP)(x; \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \sigma_{\text{tot}}(P)(x; -\xi).$$

The field \mathbf{k} . Let $\widehat{\mathbf{k}} := \mathbb{C}[[\tau^{-1}, \tau]]$ be the field of formal Laurent series in τ^{-1} . We consider the filtered subfield \mathbf{k} of $\widehat{\mathbf{k}}$ consisting of series $a = \sum_{-\infty < j \leq m} a_j \tau^j$ ($a_j \in \mathbb{C}$, $m \in \mathbb{Z}$) satisfying:

$$(1.6) \quad \text{there exists } C > 0 \text{ such that } |a_j| \leq C^{-j}(-j)! \text{ for all } j \leq 0.$$

We denote by \mathbf{k}_0 the subring of \mathbf{k} consisting of elements of order ≤ 0 and by $\mathbf{k}(r)$ the \mathbf{k}_0 -module consisting of elements of order $\leq r$.

¹An anti-isomorphism of algebras $A \xrightarrow{\sim} B$ is an isomorphism of algebras $A \xrightarrow{\sim} B^{\text{op}}$ where B^{op} is the opposite algebra.

We denote by ${}^t(\cdot): \mathbf{k} \rightarrow \mathbf{k}$ the \mathbb{C} -linear automorphism of \mathbf{k} induced by ${}^t(\tau) = -\tau$ and call it the transposition. We say that a \mathbb{C} -linear map $u: E \rightarrow F$ of \mathbf{k} -vector spaces is anti- \mathbf{k} -linear if it satisfies $u(a \cdot x) = {}^t a \cdot u(x)$ for any $x \in E$, $a \in \mathbf{k}$.

The ring \mathcal{W}_{T^*X} . On T^*X there exists a *no more conic* sheaf \mathcal{W}_{T^*X} which enjoys the following properties:

- \mathcal{W}_{T^*X} is a filtered sheaf of central \mathbf{k} -algebras and

$$\text{gr } \mathcal{W}_{T^*X} \simeq \mathcal{O}_{T^*X}[\tau^{-1}, \tau].$$

- There is a faithful and flat monomorphism of filtered \mathbb{C} -algebras $\mathcal{E}_{T^*X} \hookrightarrow \mathcal{W}_{T^*X}$.
- Set

$$(1.7) \quad \mathcal{W}_{T^*X}^{\sqrt{v}} = \pi^{-1} \Omega_X^{\frac{1}{2}} \otimes_{\pi^{-1} \mathcal{O}_X} \mathcal{W}_{T^*X} \otimes_{\pi^{-1} \mathcal{O}_X} \pi^{-1} \Omega_X^{-\frac{1}{2}}.$$

The sheaf of algebras $\mathcal{W}_{T^*X}^{\sqrt{v}}$ is endowed with an anti- \mathbf{k} -linear anti-automorphism $P \mapsto {}^t P$.

- Any symplectic isomorphism $\psi: T^*X \supset U \xrightarrow{\sim} V \subset T^*Y$ can be *locally* quantized as an isomorphism of filtered sheaves of \mathbf{k} -algebras commuting with the anti- \mathbf{k} -linear anti-isomorphism $P \mapsto {}^t P$:

$$\Psi: \psi_* \mathcal{W}_{T^*X}^{\sqrt{v}} \xrightarrow{\sim} \mathcal{W}_{T^*Y}^{\sqrt{v}}.$$

(Again, this isomorphism Ψ exists *only locally and is not unique*.)

Moreover, when X is affine, \mathcal{W}_{T^*X} satisfies:

- any section $P \in \mathcal{W}_{T^*X}(U)$ on an open subset $U \subset T^*X$ admits a total symbol

$$(1.8) \quad \sigma_{\text{tot}}(P)(x; u, \tau) = \sum_{-\infty < j \leq m} p_j(x; u) \tau^j, \quad m \in \mathbb{Z} \quad p_j \in \mathcal{O}_{T^*X}(U),$$

with the condition:

$$(1.9) \quad \begin{cases} \text{for any compact subset } K \text{ of } U \text{ there exists a positive constant } C_K \\ \text{such that } \sup_K |p_j| \leq C_K^{-j} (-j)! \text{ for all } j \leq 0. \end{cases}$$

Note that $\mathbf{k} = \mathcal{W}_{\text{pt}}$.

- The total symbol of the product is given by the Leibniz rule:

$$\sigma_{\text{tot}}(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{\tau^{-|\alpha|}}{\alpha!} \partial_u^\alpha \sigma_{\text{tot}}(P) \partial_x^\alpha \sigma_{\text{tot}}(Q).$$

- The total symbol of the transposition is given by

$$(1.10) \quad \sigma_{\text{tot}}({}^t P)(x; u, \tau) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-\tau)^{-|\alpha|}}{\alpha!} \partial_u^\alpha \partial_x^\alpha \sigma_{\text{tot}}(P)(x; u, -\tau).$$

We denote by $\mathcal{W}_{T^*X}(0)$ the subsheaf of \mathcal{W}_{T^*X} consisting of sections of order ≤ 0 . Then $\mathcal{W}_{T^*X}(0)$ is a \mathbf{k}_0 -algebra and there is a \mathbf{k} -linear isomorphism

$$\mathcal{W}_{T^*X}(0) \otimes_{\mathbf{k}_0 T^*X} \mathbf{k}_{T^*X} \xrightarrow{\sim} \mathcal{W}_{T^*X}.$$

From \mathcal{E} to \mathcal{W} . One can deduce the algebra \mathcal{W}_{T^*X} from the algebra \mathcal{E}_{T^*X} . Let $t \in \mathbb{C}$ be the coordinate and set

$$\mathcal{E}_{T^*(X \times \mathbb{C}), \hat{t}} = \{P \in \mathcal{E}_{T^*(X \times \mathbb{C})}; [P, \partial_t] = 0\}.$$

Set $T_{\tau \neq 0}^*(X \times \mathbb{C}) = \{(x, t; \xi, \tau); \tau \neq 0\}$, and consider the map

$$\begin{aligned} \rho: T_{\tau \neq 0}^*(X \times \mathbb{C}) &\rightarrow T^*X, \\ (x, t; \xi, \tau) &\mapsto (x; \xi/\tau). \end{aligned}$$

The ring \mathcal{W}_{T^*X} on T^*X may be defined by setting (see [24]):

$$\mathcal{W}_{T^*X} := \rho_*(\mathcal{E}_{T^*(X \times \mathbb{C}), \hat{t}}|_{T_{\tau \neq 0}^*(X \times \mathbb{C})}).$$

REMARK 1.1. (i) Many authors use the parameter \hbar instead of τ^{-1} .

(ii) There exist formal versions $\hat{\mathcal{E}}_{T^*X}$ and $\hat{\mathcal{W}}_{T^*X}$ of the sheaves \mathcal{E}_{T^*X} and \mathcal{W}_{T^*X} , respectively, and most of the authors work with $\hat{\mathcal{W}}_{T^*X}$.

Deformation quantization and exponential star products. If $P \in \mathcal{W}_{T^*X}$ has order 0, the operator $\exp \tau P$ does not exist in \mathcal{W}_{T^*X} . Using an extra central parameter t , a new \mathbf{k} -algebra $\mathcal{W}_{T^*X}^t$ on T^*X was constructed in [10]. This algebra enjoys the following properties:

- (i) there is a monomorphism of \mathbf{k} -algebras $\iota: \mathcal{W}_{T^*X} \hookrightarrow \mathcal{W}_{T^*X}^t$ and a morphism $\text{res}: \mathcal{W}_{T^*X}^t \rightarrow \mathcal{W}_{T^*X}$ such that the composition $\mathcal{W}_{T^*X} \rightarrow \mathcal{W}_{T^*X}^t \rightarrow \mathcal{W}_{T^*X}$ is the identity,
- (ii) any symplectic isomorphism $\psi: T^*X \supset U_X \xrightarrow{\sim} U_Y \subset T^*Y$ can be locally quantized as an isomorphism of \mathbf{k} -algebras $\Psi: \mathcal{W}_{T^*X}^t \xrightarrow{\sim} \mathcal{W}_{T^*Y}^t$,
- (iii) for $P \in \mathcal{W}_{T^*X}(0)$, the section $\exp(t\tau P) = \sum_{n \geq 0} \frac{(t\tau P)^n}{n!}$ is well defined in $\mathcal{W}_{T^*X}^t$.

The algebra $\mathcal{W}_{T^*X}^t$ is constructed as follows.

Let s be a holomorphic coordinate on \mathbb{C} and denote by $\mathcal{W}_{T^*(\mathbb{C} \times X), \hat{\partial}_s}$ the subalgebra of $\mathcal{W}_{T^*(\mathbb{C} \times X)}$ consisting of sections which do not depend on ∂_s , *i.e.*, which commute with s . We look at $\mathcal{W}_{T^*(\mathbb{C} \times X), \hat{\partial}_s}$ as a sheaf on $\mathbb{C} \times T^*X$ and we denote by $p: \mathbb{C} \times T^*X \rightarrow T^*X$ the projection. Set

$$(1.11) \quad \mathcal{W}_{T^*X}^s := R^1 p_!(\mathcal{W}_{T^*(\mathbb{C} \times X), \hat{\partial}_s}).$$

Hence, the sections of $\mathcal{W}_{T^*X}^s$ are sections of \mathcal{W}_{T^*X} depending of an extra holomorphic parameter s defined for $|s| \gg 0$ modulo sections defined for all s . The convolution product in the s variable allows us to endow $\mathcal{W}_{T^*X}^s$ with a structure of an algebra.

The algebra $\mathcal{W}_{T^*X}^t$ is constructed as the Laplace transform of $\mathcal{W}_{T^*X}^s$ which interchanges s^{-n-1} and $\frac{(t\tau)^n}{n!}$. (Here, t and τ commute.)

Note that, if P has order 0, the section $s - P$ is invertible on each compact subset K of T^*X for $|s| \gg 0$. Therefore, $1/(s - P)$ is a well-defined section of $\mathcal{W}_{T^*X}^s$ and its Laplace transform $\exp(t\tau P)$ belongs to $\mathcal{W}_{T^*X}^t$.

2. Algebroid stacks

A local model for a complex symplectic manifold is an open subset of T^*X . Hence, it is natural to ask whether the construction of the sheaf of algebras \mathcal{W}_{T^*X} still makes sense on complex symplectic manifolds. However, since the quantization

of a symplectic isomorphism is not unique, one has to replace the notion of a sheaf of algebras by that of an algebroid stack, a notion introduced in [21]. We refer to [8] for a more systematic study and to [19] for an introduction to stacks.

In this section, \mathbb{K} denotes a commutative unital algebra and X a topological space.

If A is a \mathbb{K} -algebra, we denote by A^+ the category with one object and having A as morphisms of this object. Let \mathcal{A} be a sheaf of \mathbb{K} -algebras on X and consider the prestack $U \mapsto \mathcal{A}(U)^+$ (U open in X). We denote by \mathcal{A}^+ the associated stack and call \mathcal{A}^+ the algebroid stack associated with \mathcal{A} .

Consider an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X , sheaves of \mathbb{K} -algebras \mathcal{A}_i on U_i ($i \in I$) and isomorphisms $f_{ij}: \mathcal{A}_j|_{U_{ij}} \xrightarrow{\sim} \mathcal{A}_i|_{U_{ij}}$ ($i, j \in I$). The existence of a sheaf of \mathbb{K} -algebras \mathcal{A} locally isomorphic to \mathcal{A}_i requires the condition $f_{ij}f_{jk} = f_{ik}$ on triple intersections. Let us weaken this last condition by assuming that there exist invertible sections $a_{ijk} \in \mathcal{A}_i(U_{ijk})$ satisfying

$$(2.1) \quad \begin{cases} f_{ij}f_{jk} = \text{Ad}(a_{ijk})f_{ik} \text{ on } U_{ijk}, \\ a_{ijk}a_{ikl} = f_{ij}(a_{jkl})a_{ikl} \text{ on } U_{ijkl}. \end{cases}$$

(Recall that $\text{Ad}(a)(b) = a \cdot b \cdot a^{-1}$.) One calls

$$(2.2) \quad (\{\mathcal{A}_i\}_{i \in I}, \{f_{ij}\}_{i, j \in I}, \{a_{ijk}\}_{i, j, k \in I})$$

a descent datum for \mathbb{K} -algebroid stacks on \mathcal{U} . For such a descent datum, we shall denote by $f_{ij}^+: \mathcal{A}_j^+ \xrightarrow{\sim} \mathcal{A}_i^+$ the equivalences of stacks associated with the isomorphisms f_{ij} . The following result is stated (in a different form) in [15] and goes back to [13].

THEOREM 2.1. *Consider a descent datum (2.2) on \mathcal{U} . Then there exist a stack \mathcal{A}^+ on X , equivalences of stacks $\varphi_i: \mathcal{A}^+|_{U_i} \xrightarrow{\sim} \mathcal{A}_i^+$ and isomorphisms of functors $c_{ij}: f_{ij}^+ \xrightarrow{\sim} \varphi_i \circ \varphi_j^{-1}$ satisfying $c_{ij} \circ c_{jk} \circ a_{ijk} = c_{ik}$. Moreover, the data $(\mathcal{A}^+, \{\varphi_i\}_i, \{c_{ij}\}_{ij})$ are unique up to equivalence of stacks, this equivalence being unique up to a unique isomorphism.*

One calls \mathcal{A}^+ an algebroid stack. Although \mathcal{A}^+ is not a sheaf of algebras, modules over \mathcal{A}^+ are well defined. They are described by pairs

$$\mathcal{M} = (\{\mathcal{M}_i\}_{i \in I}, \{\xi_{ij}\}_{i, j \in I}),$$

where \mathcal{M}_i is an \mathcal{A}_i -module and $\xi_{ij}: f_{ji}\mathcal{M}_j|_{U_{ij}} \rightarrow \mathcal{M}_i|_{U_{ij}}$ is an isomorphism of \mathcal{A}_i -modules such that for any $u_k \in \mathcal{M}_k$ one has

$$(2.3) \quad \xi_{ij}(f_{ji}\xi_{jk}(u_k)) = \xi_{ik}(a_{kji}^{-1}u_k).$$

Here, $f_{ji}\mathcal{M}_j$ is the \mathcal{A}_i -module deduced from the \mathcal{A}_j -module $\mathcal{M}_j|_{U_{ij}}$ by the isomorphism f_{ji} .

One gets a Grothendieck category $\text{Mod}(\mathcal{A}^+)$ and the prestack $\mathfrak{Mod}(\mathcal{A}^+)$ given by $U \mapsto \text{Mod}(\mathcal{A}^+|_U)$ is a stack equivalent on U_i to the stack $\mathfrak{Mod}(\mathcal{A}_i)$.

Twisted sheaves. As a particular case of a module over an algebroid stack, one has the notion of a twisted sheaf. Assume that \mathbb{K} is a field and denote by \mathbb{K}^\times the group of its invertible elements.

Let X be a manifold and let $\mathbf{c} \in H^2(X; \mathbb{K}^\times)$. Represent \mathbf{c} by a Čech cocycle $\{c_{ijk}\}_{i, j, k \in I}$ associated to an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X . We thus get a descent datum for \mathbb{K} -algebroid stacks

$$\mathbb{K}_{X, \mathbf{c}} := (\{\mathbb{K}_{U_i}\}_{i \in I}, \{\text{id}_{\mathbb{K}_{U_{ij}}}\}_{i, j \in I}, \{c_{ijk}\}_{i, j, k \in I}),$$

and cohomologous cocycles give equivalent stacks.

EXAMPLE 2.2. Assume now that X is a complex manifold. Consider the short exact sequence

$$1 \rightarrow \mathbb{C}_X^\times \rightarrow \mathcal{O}_X^\times \xrightarrow{d \log} d\mathcal{O}_X \rightarrow 0$$

which gives rise to the long exact sequence

$$H^1(X; \mathbb{C}_X^\times) \xrightarrow{\alpha} H^1(X; \mathcal{O}_X^\times) \xrightarrow{\beta} H^1(X; d\mathcal{O}_X) \xrightarrow{\gamma} H^2(X; \mathbb{C}_X^\times).$$

If \mathcal{L} is a line bundle, it defines a class $[\mathcal{L}] \in H^1(X; \mathcal{O}_X^\times)$. For $\lambda \in \mathbb{C}$, one sets

$$\mathbf{c}_{\mathcal{L}}^\lambda = \gamma(\lambda \cdot \beta([\mathcal{L}])) \in H^2(X; \mathbb{C}_X^\times).$$

We shall apply this construction when $\mathcal{L} = \Omega_X$ and $\lambda = \frac{1}{2}$ and set for short:

$$\text{Mod}(\mathbb{C}_{X, \frac{1}{2}}) = \text{Mod}(\mathbb{C}_{X, \mathbf{c}_{\Omega_X}^{\frac{1}{2}}}).$$

3. Quantization of symplectic manifolds

On any complex contact manifold, the existence of a canonical \mathbb{C} -algebroid stack locally equivalent to the algebroid stack associated with the sheaf of algebras of microdifferential operators of [25] has been obtained by M. Kashiwara in [15].

On any complex Poisson manifold, the existence of a $\widehat{\mathbf{k}}$ -algebroid stack of formal deformation quantization has been obtained by M. Kontsevich [21]. The analytic case on symplectic manifolds has been obtained in [24] by a different method, making a link with Kashiwara's construction. The classification of these algebroid stacks is discussed in [23].

In particular, for a complex symplectic manifold \mathfrak{X} , there is a canonical \mathbf{k} -algebroid stack $\mathcal{W}_{\mathfrak{X}}^{\sqrt{v}, +}$ locally equivalent to the algebroid stack $\mathcal{W}_{T^*X}^{\sqrt{v}, +}$ associated with the sheaf of algebras $\mathcal{W}_{T^*X}^{\sqrt{v}}$. The same result holds with $\mathcal{W}_{\mathfrak{X}}$ replaced by $\mathcal{W}_{\mathfrak{X}}(0)$ and \mathbf{k} by \mathbf{k}_0 .

NOTATION 3.1. For short, as far as there is no risk of confusion, we shall write $\mathcal{W}_{\mathfrak{X}}$ instead of $\mathcal{W}_{\mathfrak{X}}^{\sqrt{v}, +}$.

Let \mathfrak{X} be a complex symplectic manifold. Then $\text{Mod}(\mathcal{W}_{\mathfrak{X}})$ is a Grothendieck category. We denote by $\text{D}^b(\mathcal{W}_{\mathfrak{X}})$ its bounded derived category and call an object of this derived category a $\mathcal{W}_{\mathfrak{X}}$ -module. One proves as usual that the sheaf of algebras \mathcal{W}_{T^*X} is coherent and the support of a coherent \mathcal{W}_{T^*X} -module is a closed complex analytic subvariety of T^*X . This support is involutive in view of Gabber's theorem (see [16, Th. 7.33]). Hence, the (local) notions of a coherent or holonomic $\mathcal{W}_{\mathfrak{X}}$ -module make sense.

Similarly as for \mathcal{D} -modules (see [16]), one says that a coherent $\mathcal{W}_{\mathfrak{X}}$ -module \mathcal{M} is good if, for any open relatively compact subset U of \mathfrak{X} , there exists a coherent $\mathcal{W}_{\mathfrak{X}}(0)|_U$ -module \mathcal{M}_0 contained in $\mathcal{M}|_U$ which generates $\mathcal{M}|_U$.

Let us denote by:

- $\text{D}_{\text{coh}}^b(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $\text{D}^b(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with coherent cohomologies.
- $\text{D}_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $\text{D}_{\text{coh}}^b(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with good cohomologies.

- $D_{\text{gd},c}^b(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $D_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with compact supports.
- $D_{\text{hol}}^b(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $D_{\text{coh}}^b(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with Lagrangian supports in \mathfrak{X} . (One calls such an object an holonomic $\mathcal{W}_{\mathfrak{X}}$ -module.)
- $D_{\text{rh}}^b(\mathcal{W}_{\mathfrak{X}})$ the full triangulated subcategory of $D_{\text{hol}}^b(\mathcal{W}_{\mathfrak{X}})$ consisting of objects with regular holonomic cohomologies (to be defined below).
- Let \mathfrak{X} and \mathfrak{Y} be two complex symplectic manifolds and let $\mathcal{M} \in D^b(\mathcal{W}_{\mathfrak{X}})$, $\mathcal{N} \in D^b(\mathcal{W}_{\mathfrak{Y}})$. Their exterior product is given by $\mathcal{M} \boxtimes \mathcal{N} := \mathcal{W}_{\mathfrak{X} \times \mathfrak{Y}} \boxtimes_{\mathcal{W}_{\mathfrak{X}} \boxtimes \mathcal{W}_{\mathfrak{Y}}} (\mathcal{M} \boxtimes \mathcal{N})$.

Simple $\mathcal{W}_{\mathfrak{X}}$ -modules.

DEFINITION 3.2. Let Λ be a smooth Lagrangian submanifold of \mathfrak{X} .

- Let $\mathcal{L}(0)$ be a coherent $\mathcal{W}_{\mathfrak{X}}(0)$ -module supported by Λ . One says that $\mathcal{L}(0)$ is simple along Λ if $\mathcal{L}(0)/\mathcal{L}(-1)$ is an invertible \mathcal{O}_{Λ} -module. Here, $\mathcal{L}(-1) = \mathbf{k}_{\mathfrak{X}}(-1)\mathcal{L}(0)$.
- Let \mathcal{L} be a coherent $\mathcal{W}_{\mathfrak{X}}$ -module supported by Λ . One says that \mathcal{L} is simple along Λ if there locally exists a coherent $\mathcal{W}_{\mathfrak{X}}(0)$ -submodule $\mathcal{L}(0)$ of \mathcal{L} such that $\mathcal{L}(0)$ generates \mathcal{L} over $\mathcal{W}_{\mathfrak{X}}$ and is simple along Λ .
- Let \mathcal{L} be a coherent $\mathcal{W}_{\mathfrak{X}}$ -module supported by Λ . One says that \mathcal{L} is regular if, locally, it is a finite direct sum of simple modules.
- Let Λ be a, not necessarily smooth, Lagrangian subvariety of \mathfrak{X} . A coherent $\mathcal{W}_{\mathfrak{X}}$ -module supported by Λ is regular if it is regular at generic points of Λ . One calls such an object a regular holonomic $\mathcal{W}_{\mathfrak{X}}$ -module.

It follows from Gabber's theorem that when Λ is smooth, Definitions 3.2 (c) and (d) coincide (see [16, Th. 8.34]).

One proves easily that any two $\mathcal{W}_{\mathfrak{X}}$ -modules simple along Λ are locally isomorphic and that if \mathcal{L}_i ($i = 0, 1$) are simple along Λ , then $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1)$ is concentrated in degree 0 and is a \mathbf{k} -local system of rank one on Λ .

EXAMPLE 3.3. Let X be a complex manifold. We denote by \mathcal{O}_X^{τ} the \mathcal{W}_{T^*X} -module supported by the zero-section T_X^*X defined by $\mathcal{O}_X^{\tau} = \mathcal{W}_{T^*X}/\mathcal{I}$, where \mathcal{I} is the left ideal generated by the vector fields which annihilate the section $1 \in \mathcal{O}_X$. A section $f(x, \tau)$ of this module may be written as a series:

$$(3.1) \quad f(x, \tau) = \sum_{-\infty < j \leq m} f_j(x) \tau^j, \quad m \in \mathbb{Z},$$

the f_j 's satisfying Condition (1.9). Then \mathcal{O}_X^{τ} is a simple \mathcal{W}_{T^*X} -module along T_X^*X .

The next result asserts that, up to a twist, there exist globally defined simple $\mathcal{W}_{\mathfrak{X}}$ -modules.

THEOREM 3.4. [9] *Let Λ be a smooth Lagrangian submanifold of \mathfrak{X} . There is an equivalence of \mathbf{k} -additive stacks:*

$$(3.2) \quad \mathfrak{Mod}_{\text{reg-}\Lambda}(\mathcal{W}_{\mathfrak{X}})|_{\Lambda} \simeq \mathfrak{Mod}_{\text{loc-sys}}(\mathbf{k}_{\Lambda} \otimes_{\mathbb{C}} \mathbb{C}_{\Lambda, 1/2}).$$

Here, the left-hand side is the substack of $\mathfrak{Mod}(\mathcal{W}_{\mathfrak{X}})|_{\Lambda}$ consisting of regular holonomic modules along Λ and the right-hand side is the substack of the stack of twisted sheaves of \mathbf{k}_{Λ} -modules with twist $\mathbb{C}_{\Lambda, 1/2}$ consisting of objects locally isomorphic to local systems over \mathbf{k} . The proof uses the corresponding theorem for contact manifolds due to Kashiwara.

$\mathcal{W}_{\mathfrak{X}}$ -module associated with the diagonal. Let \mathfrak{X} be a complex symplectic manifold. We denote by \mathfrak{X}^a the complex manifold \mathfrak{X} endowed with the symplectic form $-\omega$, where ω is the symplectic form on \mathfrak{X} . There is a natural equivalence of algebroid stacks $\mathcal{W}_{\mathfrak{X}}^{\text{op}} \simeq \mathcal{W}_{\mathfrak{X}^a}$.

We denote by $\Delta_{\mathfrak{X}}$ the diagonal of $\mathfrak{X} \times \mathfrak{X}^a$ and by $d_{\mathfrak{X}}$ the complex dimension of \mathfrak{X} .

THEOREM 3.5. (i) *There exists a simple $\mathcal{W}_{\mathfrak{X} \times \mathfrak{X}^a}$ -module $\mathcal{C}_{\Delta_{\mathfrak{X}}}$ supported by the diagonal $\Delta_{\mathfrak{X}}$ of $\mathfrak{X} \times \mathfrak{X}^a$ with the property that if U is open in \mathfrak{X} and isomorphic to an open subset V of a cotangent bundle T^*X , then $\mathcal{C}_{\Delta_{\mathfrak{X}}}|_U$ is isomorphic to $\mathcal{W}_{T^*X}|_V$ as a $\mathcal{W}_{T^*X} \otimes \mathcal{W}_{T^*X}^{\text{op}}$ -module.*

(ii) *There is a natural isomorphism $\mathcal{C}_{\Delta_{\mathfrak{X}^a}} \overset{\text{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X} \times \mathfrak{X}^a}} \mathcal{C}_{\Delta_{\mathfrak{X}}} \simeq \mathbf{k}_{\Delta_{\mathfrak{X}}}[d_{\mathfrak{X}}]$.*

(i) follows from general considerations on algebroid stacks. (ii) follows from a construction of Feigin and Tsygan [12] (see also [11]).

Let $\mathcal{M} \in \text{D}^b(\mathcal{W}_{\mathfrak{X}})$. We set

$$(3.3) \quad \text{D}'_{\text{w}}\mathcal{M} := R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{C}_{\Delta_{\mathfrak{X}}}), \quad \text{D}_{\text{w}}\mathcal{M} := \text{D}'_{\text{w}}\mathcal{M}[\tfrac{1}{2}d_{\mathfrak{X}}].$$

These objects are well-defined in $\text{D}^b(\mathcal{W}_{\mathfrak{X}^a})$. Using Theorem 3.5 and the fact that $\mathcal{C}_{\Delta_{\mathfrak{X}}}$ is simple, one gets an isomorphism in $\text{D}_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}^a \times \mathfrak{X}})$:

$$(3.4) \quad \text{D}_{\text{w}}(\mathcal{C}_{\Delta_{\mathfrak{X}}}) \simeq \mathcal{C}_{\Delta_{\mathfrak{X}^a}}.$$

REMARK 3.6. By extending the definition of Van den Bergh [29] (see also [32]) to algebroid stacks, the isomorphism (3.4) could be translated by saying that $\mathcal{C}_{\Delta_{\mathfrak{X}}}[d_{\mathfrak{X}}]$ is a rigid dualizing complex over $\mathcal{W}_{\mathfrak{X}}$.

Let \mathcal{M}, \mathcal{N} be two objects of $\text{D}_{\text{coh}}^b(\mathcal{W}_{\mathfrak{X}})$. Then, after identifying $\Delta_{\mathfrak{X}}$ with \mathfrak{X} by the first projection, there is a natural isomorphism in $\text{D}^b(\mathbf{k}_{\mathfrak{X}})$:

$$R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{N}) \simeq R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X} \times \mathfrak{X}^a}}(\mathcal{M} \boxtimes \text{D}'_{\text{w}}\mathcal{N}, \mathcal{C}_{\Delta_{\mathfrak{X}}}).$$

4. Constructibility and perversity

We refer to [18] for basic notions on sheaves. Let Z be a real analytic manifold. Recall that one denotes by:

- $C(S_1, S_2)$ the normal cone of two subsets S_1 and S_2 of Z , a closed conic subset of the tangent space TZ , identified with a subset of T^*Z in case Z is symplectic,
- $\text{D}^b(\mathbf{k}_Z)$ the bounded derived category of sheaves of \mathbf{k} -modules on Z ,
- $\text{SS}(F)$ the microsupport of an object $F \in \text{D}^b(\mathbf{k}_Z)$, a closed conic involutive subset of T^*Z ,
- or_Z the orientation sheaf on Z , ω_Z the dualizing complex (hence, $\omega_Z \simeq \text{or}_Z[\dim_{\mathbb{R}} Z]$), $\text{D}_Z := R\mathcal{H}om_{\mathbf{k}_Z}(\cdot, \omega_Z)$ the duality functor for sheaves,
- $\text{D}_{\mathbb{R}\text{c}}^b(\mathbf{k}_Z)$ the full triangulated subcategory of $\text{D}^b(\mathbf{k}_Z)$ consisting of objects with \mathbb{R} -constructible cohomology and, in case Z is a complex manifold, $\text{D}_{\mathbb{C}\text{c}}^b(\mathbf{k}_Z)$ the full triangulated subcategory of $\text{D}^b(\mathbf{k}_Z)$ consisting of objects with \mathbb{C} -constructible cohomology.

THEOREM 4.1. [20] *Let \mathfrak{X} be a complex symplectic manifold and let \mathcal{L}_i ($i = 0, 1$) be two objects of $\text{D}_{\text{rh}}^b(\mathcal{W}_{\mathfrak{X}})$ supported by smooth Lagrangian manifolds Λ_i . Then*

- (i) the object $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0)$ belongs to $D_{\mathbb{C}\mathfrak{c}}^b(\mathbf{k}_{\mathfrak{X}})$ and its microsupport is contained in the normal cone $C(\Lambda_0, \Lambda_1)$,
- (ii) the natural morphism

$$R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0) \rightarrow D_{\mathfrak{X}}(R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_0, \mathcal{L}_1[d_{\mathfrak{X}}]))$$

is an isomorphism.

The proof makes use of tools from the theory of holonomic \mathcal{D} -modules and uses some functional analysis, namely Houzel's theorem [14, 2, § 4 Th. 1'].

REMARK 4.2. In [2], K. Behrend and B. Fantecchi construct complexes (over \mathbb{C}) naturally associated with the data of two smooth Lagrangian submanifolds. Their result should have some relations with Theorem 4.1.

COROLLARY 4.3. *Let \mathcal{L}_0 and \mathcal{L}_1 be two regular holonomic $\mathcal{W}_{\mathfrak{X}}$ -modules supported by smooth Lagrangian manifolds. Then the object $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}_1, \mathcal{L}_0)$ of $D_{\mathbb{C}\mathfrak{c}}^b(\mathbf{k}_{\mathfrak{X}})$ is perverse.*

CONJECTURE 4.4. [20] Theorem 4.1 remains true without assuming that the Λ_i 's are smooth.

5. Composition of kernels and Calabi-Yau categories

Consider three complex symplectic manifolds \mathfrak{X}_i ($i = 1, 2, 3$) and denote as usual by p_i and p_{ji} the projections defined on $\mathfrak{X}_3 \times \mathfrak{X}_2 \times \mathfrak{X}_1$.

For Λ_i a closed subset of $\mathfrak{X}_{i+1} \times \mathfrak{X}_i$ ($i = 1, 2$), we set

$$(5.1) \quad \Lambda_2 \circ \Lambda_1 := p_{31}(p_{32}^{-1}\Lambda_2 \cap p_{21}^{-1}\Lambda_1).$$

For $\mathcal{K}_i \in D_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}_{i+1} \times \mathfrak{X}_i^a})$ ($1 \leq i \leq 2$), we set

$$(5.2) \quad \mathcal{K}_2 \circ \mathcal{K}_1 := R p_{31!}(p_{32}^{-1}\mathcal{K}_2 \overset{\text{L}}{\otimes}_{p_2^{-1}\mathcal{W}_{\mathfrak{X}_2}} p_{21}^{-1}\mathcal{K}_1).$$

THEOREM 5.1. [28] *Assume that p_{31} is proper on $p_{32}^{-1}\text{supp}(\mathcal{K}_2) \cap p_{12}^{-1}\text{supp}(\mathcal{K}_1)$. Then*

- (i) the object $\mathcal{K}_2 \circ \mathcal{K}_1$ belongs to $D_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}_3 \times \mathfrak{X}_1^a})$,
- (ii) there is a natural isomorphism in $D_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}_3^a \times \mathfrak{X}_1})$:

$$(5.3) \quad D_{\text{w}}\mathcal{K}_2 \circ D_{\text{w}}\mathcal{K}_1 \xrightarrow{\sim} D_{\text{w}}(\mathcal{K}_2 \circ \mathcal{K}_1).$$

The proof of (i) uses again [14, 2, § 4 Th. 1']. The construction of the duality morphism in (ii) uses the isomorphism (3.4).

Choosing $\mathfrak{X}_3 = \mathfrak{X}_1 = \{\text{pt}\}$ in Theorem 5.1, we get:

COROLLARY 5.2. *Let \mathfrak{X} be a complex symplectic manifold and let \mathcal{M} and \mathcal{N} be two objects of $D_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}})$. Assume that $\text{supp}(\mathcal{M}) \cap \text{supp}(\mathcal{N})$ is compact. Then*

- (i) the object $R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{N})$ has \mathbf{k} -finite dimensional cohomology,
- (ii) there is a natural isomorphism, functorial with respect to \mathcal{M} and \mathcal{N} :

$$R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{N}) \simeq (R\mathcal{H}om_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{N}, \mathcal{M}[d_{\mathfrak{X}}]))^*,$$

where \star is the duality functor for \mathbf{k} -vector spaces.

Recall [3] that a \mathbf{k} -triangulated category \mathcal{T} is *Ext*-finite if for any two objects F, G of \mathcal{T} , the \mathbf{k} -vector space $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(F, G[i])$ is finite dimensional. In this situation, a Serre functor $S: \mathcal{T} \rightarrow \mathcal{T}$ is an equivalence of \mathbf{k} -triangulated categories such that

$$(\text{Hom}_{\mathcal{T}}(F, G))^* \simeq \text{Hom}_{\mathcal{T}}(G, S(F))$$

functorially in F and G . If the Serre functor is a shift by an integer d , one says that \mathcal{T} is a Calabi-Yau category of dimension d .

COROLLARY 5.3. *Let \mathfrak{X} be a complex symplectic manifold. Then $\text{D}_{\text{gd},c}^b(\mathcal{W}_{\mathfrak{X}})$ is a Calabi-Yau category of dimension $d_{\mathfrak{X}}$.*

REMARK 5.4. (i) The analogue of Corollary 5.2 on complex contact manifolds over the field \mathbb{C} is false. Note that Corollary 5.2 may be considered as a direct image theorem over one point, and a manifold of dimension 0 has a complex symplectic structure, not a complex contact structure.

Nevertheless, the analogue of Corollary 5.2 is true over the field \mathbb{C} for complex contact manifolds when restricting to the category of regular holonomic modules. This follows from results obtained in [17].

(ii) On a complex compact manifold X , the bounded derived category of sheaves with \mathbb{C} -constructible cohomology is an *Ext*-finite triangulated category, as well as the equivalent category, the bounded derived category of \mathcal{D}_X -modules with regular holonomic cohomology. Both categories do not seem to have a Serre functor.

6. Index theorem

In this section, we announce works in progress with J-P. Schneiders [28].

Euler class. Let \mathfrak{X} be complex symplectic manifold and let $\mathcal{M} \in \text{D}_{\text{coh}}^b(\mathcal{W}_{\mathfrak{X}})$. We have the chain of morphisms

$$\begin{aligned} R\text{Hom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{M}) &\xleftarrow{\sim} R\text{Hom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{C}_{\Delta_{\mathfrak{X}}}) \overset{\text{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X}}} \mathcal{M} \\ &\simeq (R\text{Hom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{C}_{\Delta_{\mathfrak{X}}}) \otimes_{\mathbf{k}_{\mathfrak{X}}} \mathcal{M}) \overset{\text{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X}} \otimes \mathcal{W}_{\mathfrak{X}^a}} \mathcal{C}_{\Delta_{\mathfrak{X}}} \\ &\rightarrow \mathcal{C}_{\Delta_{\mathfrak{X}^a}} \overset{\text{L}}{\otimes}_{\mathcal{W}_{\mathfrak{X}} \times \mathfrak{X}^a} \mathcal{C}_{\Delta_{\mathfrak{X}}} \simeq \mathbf{k}_{\mathfrak{X}}[d_{\mathfrak{X}}]. \end{aligned}$$

Here, we have used Theorem 3.5 (ii). We get a map

$$\text{Hom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{M}) \rightarrow H_{\text{supp}(\mathcal{M})}^{d_{\mathfrak{X}}}(\mathfrak{X}; \mathbf{k}_{\mathfrak{X}}).$$

The image of $\text{id}_{\mathcal{M}}$ gives an element

$$(6.1) \quad \text{Eu}(\mathcal{M}) \in H_{\text{supp}(\mathcal{M})}^{d_{\mathfrak{X}}}(\mathfrak{X}; \mathbf{k}_{\mathfrak{X}}).$$

Symplectic Riemann-Roch theorem. We consider the situation of Theorem 5.1. Hence, we have three complex symplectic manifolds \mathfrak{X}_i ($i = 1, 2, 3$) and we have closed subsets Λ_i of $\mathfrak{X}_{i+1} \times \mathfrak{X}_i$ ($i = 1, 2$). We set for short $d_i := d_{\mathfrak{X}_i}$ ($i = 1, 2, 3$) and consider cohomology classes $\lambda_i \in H_{\Lambda_{i+1} \times \Lambda_i}^{d_{i+1} + d_i}(\mathfrak{X}_{i+1} \times \mathfrak{X}_i; \mathbf{k}_{\mathfrak{X}_{i+1} \times \mathfrak{X}_i})$ ($i = 1, 2$). Assuming that p_{31} is proper on $p_{32}^{-1}(\Lambda_2) \cap p_{12}^{-1}(\Lambda_1)$, we set

$$\lambda_2 \circ \lambda_1 := \int_{\mathfrak{X}_2} (p_{32}^{-1} \lambda_2 \cup p_{21}^{-1} \lambda_1) \in H_{\Lambda_2 \circ \Lambda_1}^{d_3 + d_1}(\mathfrak{X}_3 \times \mathfrak{X}_1; \mathbf{k}_{\mathfrak{X}_3 \times \mathfrak{X}_1}).$$

Here, \cup is the cup product and

$$\int_{\mathfrak{X}_2} : H_{p_{32}^{-1}\Lambda_2 \cap p_{21}^{-1}\Lambda_1}^{d_3+2d_2+d_1}(\mathfrak{X}_3 \times \mathfrak{X}_2 \times \mathfrak{X}_1; \mathbf{k}_{\mathfrak{X}_3 \times \mathfrak{X}_2 \times \mathfrak{X}_1}) \rightarrow H_{\Lambda_2 \circ \Lambda_1}^{d_3+d_1}(\mathfrak{X}_3 \times \mathfrak{X}_1; \mathbf{k}_{\mathfrak{X}_3 \times \mathfrak{X}_1})$$

is the Poincaré integration morphism.

Let $\mathcal{K}_i \in D_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}_{i+1} \times \mathfrak{X}_i^a})$ ($1 \leq i \leq 2$) and assume that p_{31} is proper on $p_{32}^{-1} \text{supp}(\mathcal{K}_2) \cap p_{12}^{-1} \text{supp}(\mathcal{K}_1)$. The proof of the formula

$$(6.2) \quad \text{Eu}(\mathcal{K}_2 \circ \mathcal{K}_1) = \text{Eu}(\mathcal{K}_2) \circ \text{Eu}(\mathcal{K}_1)$$

is in progress. It would partly generalize the index theorems for coherent \mathcal{D}_X -modules proved in [27].

As a particular case of (6.2), one finds that for two objects \mathcal{L} and \mathcal{M} in $D_{\text{gd}}^b(\mathcal{W}_{\mathfrak{X}})$ such that $\text{supp } \mathcal{L} \cap \text{supp } \mathcal{M}$ is compact, we have

$$(6.3) \quad \chi(\text{RHom}_{\mathcal{W}_{\mathfrak{X}}}(\mathcal{L}, \mathcal{M})) = \int_{\mathfrak{X}} \text{Eu}(D'_w \mathcal{L}) \cup \text{Eu}(\mathcal{M}).$$

In the case of coherent \mathcal{D}_X -modules on a complex manifold X , the formula

$$\text{Eu}(\mathcal{L}) = [\text{Ch}(\text{gr } \mathcal{L}) \cup \text{Td}(TX)]^{d_{T^*X}}$$

had been conjectured in [27] and proved by [6]. On a complex symplectic manifold, these authors give a more general formula calculating $\text{Eu}(\mathcal{L})$, a formula in which the class of the deformation quantization appears.

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